Ex 1 Does $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converge or diverge?

(a) Converges

(b) Diverges

$2^n = \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}}$

$n! = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}_{n \text{ things}}$

$0 \le \frac{2^n}{n!} = \frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{2} \cdot \frac{2}{1} \le \frac{2 \cdot 2 \cdot 2}{n(n-1)} = \frac{8}{n(n-1)}$

(≤ 1 for $\frac{2}{n-2}$ and ≤ 1 for $\frac{2}{2}$)

Note $\sum_{n=1}^{\infty} \frac{8}{n(n-1)}$ converges (use L.C.T.), so by the D.C.T. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

Q. Why can't we use the integral test to determine the behavior of the above series?

No such function $\frac{2^x}{x!}$ because $n!$ is only defined for integers n.

Q. How do we work with factorials?

r - constant

Q. Consider the sequence defined by $a_n = r^n$ for $n \geqslant 0$ and the corresponding series $\sum_{n=0}^{\infty} a_n$.

(a) Under what conditions does the series converge?

$\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$

(b) What is the ratio $\frac{a_{n+1}}{a_n}$?

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r$$

(c) Rewrite the conditions from (a) to make a statement like "$\sum_{n=0}^{\infty} a_n$ converges if and only if [insert-something-about-$\frac{a_{n+1}}{a_n}$-here]"

$\sum_{n=0}^{\infty} a_n$ converges if and only if $\left|\frac{a_{n+1}}{a_n}\right| < 1$

↑ only geometric series

Q. What is $\frac{(n+1)!}{n!}$?

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cdot n\cdot (n-1)\cdots 2\cdot 1}{n\cdot (n-1)\cdots 2\cdot 1} = n+1$$

nice cts.

Thm: (The Ratio Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with nonzero terms and suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r$.

Then

(a) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

(b) If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ converge.

(c) If $r = 1$, then the test is inconclusive.

Ex 2 Determine the behavior of $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$.

$$a_n = \frac{(n!)^3}{(3n)!}, \quad a_{n+1} = \frac{((n+1)!)^3}{(3(n+1))!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left\{\frac{((n+1)!)^3}{(3(n+1))!}\right\}}{\left\{\frac{(n!)^3}{(3n)!}\right\}}$$

$$= \lim_{n\to\infty}\frac{(3n)!}{(3(n+1))!}\,\frac{((n+1)!)^3}{(n!)^3}$$

$$= \lim_{n\to\infty}\frac{(3n)(3n-1)(3n-2)\cdots 2\cdot 1}{(3n+3)(3n+2)(3n+1)\cdot 3n\cdot (3n-1)\cdots 2\cdot 1}\cdot\left(\frac{(n+1)!}{n!}\right)^3$$

$$= \lim_{n\to\infty}\frac{1}{(3n+3)(3n+2)(3n+1)}\cdot (n+1)^3$$

$$= \lim_{n\to\infty}\frac{n^3 + \text{lower-degree terms}}{27n^3 + \text{lower degree terms}} = \frac{1}{27} < 1$$

By the ratio test, $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ converges.

$\frac{x^a}{x^b} = x^{a-b} = \frac{1}{x^{b-a}}$ $\qquad \frac{x^3}{x^2} = x^1 = \frac{1}{x^{-1}}$

Ex 3 Use the ratio test to determine the behavior of $\sum_{n=1}^{\infty} \frac{2n^3+7n-50}{(-2)^n}$.

(a) Converges

(b) Diverges

RaT: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\left(\frac{2(n+1)^3+7(n+1)-50}{(-2)^{n+1}}\right)}{\left(\frac{2n^3+7n-50}{(-2)^n}\right)} \right|$

$= \lim_{n\to\infty} \left| \frac{2(n+1)^3+7(n+1)-50}{2n^3+7n-50} \cdot \frac{(-2)^n}{(-2)^{n+1}} \right|$

lower degree terms

$= \lim_{n\to\infty} \left| \frac{2n^3 + \text{L.D.T.}}{2n^3 + \text{L.D.T.}} \cdot \frac{1}{(-2)^{n+1-n}} \right|$ → b/c same leading term in num. and denom.

$= \left| 1 \cdot \frac{-1}{2} \right| = \frac{1}{2} < 1$

By the ratio test, $\sum_{n=1}^{\infty} \frac{2n^3+7n-50}{(-2)^n}$ Converges

Q. For what types of series is the ratio test helpful?

- factorials
- (constant)$^{\text{fn. of } n}$ (function) e.g. 5^{n^2-n}
- last resort?

Q. Why not another test for

$$\sum_{n=1}^{\infty} \frac{2n^3 + 7n - 5^n}{(-2)^n} \; ?$$

A. Integral test: no pos. terms
bad to integrate

Comparison tests: no pos. terms

alt. series test ← A. S. T.: hard to show that $|\text{terms}| \longrightarrow 0$ monotonically

abs. conv. test ← A. C. T: the requires another test

Ex 4 What happens if we try to use the ratio test on p-series?

Recall: a p-series is $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{1}{(n+1)^p}\right)}{\left(\frac{1}{n^p}\right)} \right|$$

$$= \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p$$

uses fact that x^p is cts. and p is constant

$$= \left(\lim_{n\to\infty} \frac{n}{n+1} \right)^p = 1^p = 1$$

$\Longrightarrow$ No conclusion

Ex 5 Does $\sum_{n=1}^{\infty} \frac{n^{3n}}{(2n+7)^n}$ converge or diverge?

(a) Converges

(b) Diverges

R.T.: $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^{3(n+1)}}{(2(n+1)+7)^{n+1}} \cdot \frac{(2n+7)^n}{n^{3n}}$$

$$= \lim_{n\to\infty} \frac{(2n+7)^n}{(2n+9)^{n+1}} \cdot \frac{(n+1)^{3(n+1)}}{n^{3n}}$$

Q. Consider the sequence defined by $a_n = r^n$ for $n \geqslant 0$ and the corresponding series $\sum_{n=0}^{\infty} a_n$.

(a) Under what conditions does the series converge?

$|r| < 1$

(b) What is the root $\sqrt[n]{a_n} = |a_n|^{1/n}$? (and why is the absolute value necessary?)

$|a_n|^{1/n} = |r^n|^{1/n} = |r|^{n/n} = |r|$

(c) Rewrite the conditions from (a) to make a statement like "$\sum_{n=0}^{\infty} a_n$ converges if and only if [insert-something-about-$|a_n|^{1/n}$-here]"

$\sum_{n=0}^{\infty} a_n$ converges if $|a_n|^{1/n} < 1$

not a general rule, just true for geometric series

Thm: (The Root Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with $\lim_{n\to\infty} |a_n|^{1/n} = r$. Then

(a) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(c) If $r = 1$, then the test is inconclusive.

Ex 6 Determine the behavior of $\sum_{n=1}^{\infty} \frac{n^{3n}}{(2n+7)^n}$.

a) converges

b) diverges

R. T.: $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{n^{3n}}{(2n+7)^n} \right)^{1/n}$

$= \lim_{n\to\infty} \frac{\left(n^{3n}\right)^{1/n}}{\left\{(2n+7)^n\right\}^{1/n}} = \lim_{n\to\infty} \frac{n^3}{2n+7}$

$= \infty > 1$

By the root test, $\sum_{n=1}^{\infty} \frac{n^{3n}}{(2n+7)^n}$ diverges

Ex 7 Determine the behavior of $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

(a) Converges
(b) Diverges

R.T. $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^{n^2}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$

$= \frac{1}{e} < 1$

By the root test, $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges.

Ex 8 Determine the behavior of $\sum_{n=2}^{\infty} (\ln(n))^n$.

(a) Converges
(b) Diverges

$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|(\ln(n)^n)^{1/n}\right| = \lim_{n\to\infty} \ln(n) = \infty$

By the root test, $\sum_{n=1}^{\infty} (\ln(n))^n$ diverges

Q. For what types of series is the root test helpful?

$(\text{fn. of } n)^{c\cdot n}$ where c is constant

Fact: $\lim_{n\to\infty} n^{1/n} = 1$

Why?

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \left(e^{\ln(n)}\right)^{1/n} = \lim_{n\to\infty} e^{\frac{\ln(n)}{n}}$$

$$= e^{\lim_{n\to\infty} \frac{\ln(n)}{n}}$$

since e^x is a continuous function

$$\lim_{n\to\infty} \frac{\ln(n)}{n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{1/n}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

So $\lim_{n\to\infty} n^{1/n} = e^{\lim_{n\to\infty} \frac{\ln(n)}{n}} = e^0 = 1$ ✓